Numerical experiments carried out with a wide class of functions showed results similar to those above and confirmed the reliability and effectiveness of the method.

NOTATION

Δ, Laplacian operator; $\Delta u = \partial^2 u/\partial x^2 + \partial^2 u/\partial y^2$; Γ, boundary of the rectangle.

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FINITE-ELEMENT CALCULATIONS ON NONSTATIONARY HEAT TRANSFER

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A finite-element technique has been used in solving a boundary-value problem for a two-dimensional nonstationary turbulent-diffusion equation.

The deposition and transport of particles in a flow of liquid can be described by a turbulent-diffusion equation if the concentration of the solid is low and the particles are sufficiently small. Rose [1] has defined the limits to the application of the diffusion theory with regard to particle size by experiment.

The models of [2,3] are relevant to the description of these processes, and some features of these are used here. The model of [3] describes the steady-state deposition of a solid material in a planar semiinfinite channel in the form of a boundary-value problem for a stationary equation in turbulent diffusion. A numerical solution was obtained by finite-difference methods and this is compared with experiment. Other studies [4-6] deal with models for water quality, in which the equations of hydrodynamics and turbulent diffusion are employed.

There are also other discussions [7-9] of nonstationary equations for turbulent diffusion; it has been suggested [8,9] that Galerkin's method should be used together with the finite-element technique, and the relevant systems of equations have been derived, but numerical treatments have been given only for the one-dimensional case [9] and for the two-dimensional case but neglecting convective terms [8]. In [7] we find a solution to a two-dimensional boundary-value problem subject to homogeneous Dirichlet conditions on the assumption that the turbulent-diffusion coefficients are constants and that there is a source of the minor component within the region only at the start.

Here we consider a model for the transport and deposition of a material suspended in a planar flow; we assume that the velocity components and the turbulent-diffusion coefficients are known functions of time and the coordinates, in which case the model can be represented as a boundary-value problem:

$$\frac{\partial c'}{\partial t'} + U(x, z, t')\frac{\partial c'}{\partial x} + W(x, z, t')\frac{\partial c'}{\partial z} + \omega'\frac{\partial c'}{\partial z} =$$

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$$= \frac{\partial}{\partial x} D_x(x, z, t') \frac{\partial c'}{\partial x} + \frac{\partial}{\partial z} D_z(x, z, t') \frac{\partial c'}{\partial z} + F'(x, z, t'), \ (x, z, t') \in \Omega_1 \cap \{t' > 0\}, \tag{1}$$

$$\Omega_1 = \{(x, z): 0 < x < L, \quad H(x) < z < 0\},$$

$$\Gamma_1: c'(0, z, t') = c_0(z, t'), \ H(0) \leqslant z \leqslant 0, \tag{2}$$

$$\Gamma_2: D_z \frac{\partial c'(x, z, t')}{\partial z}\bigg|_{z=0} = \omega' c'(x, 0, t'), \quad 0 \leqslant x \leqslant L, \tag{3}$$

$$\Gamma_3: D_x \frac{\partial c'(x, z, t')}{\partial x}\bigg|_{x=L} = 0, \quad H(L) \leqslant z \leqslant 0, \tag{4}$$

$$\Gamma_4: \left[D_x \frac{\partial c'}{\partial x} \cos(n, 0x) + D_z \frac{\partial c'}{\partial z} \cos(n, 0z)\right]\bigg|_{z=H(x)} = (1-\alpha)\,\omega' c'(x, H(x), t') \tag{5}$$

subject to the initial conditions

$$c'(x, z, 0) = \psi'(x, z) \tag{6}$$

(the origin is taken on the free surface, while the 0x axis is directed along the main flow, and 0z is upwards).

For simplicity, we consider the problem of (1)-(6) for a planar channel with $H(x) \equiv$ const = $-H$; then W = 0 and the distribution of the velocity U may be taken as logarithmic or some other empirical form [14].

We introduce the scale factors $U = U_m U_1$, $\omega' = U_m \omega$, $x = Lx_1$, $z = Lx_2$, $D_x = D_1 L U_*$, $D_z = D_2 L U_*$, $c' = c_1 c$, $t' = (L/U_m)t$ to reduce the equations of (1)-(6) to dimensionless form:

$$\frac{\partial c}{\partial t} + U_1(x_1, x_2, t)\frac{\partial c}{\partial x_1} + \omega \frac{\partial c}{\partial x_2} = \beta \sum_{l=1}^{2} \frac{\partial}{\partial x_l} D_l(x_1, x_2, t) \frac{\partial c}{\partial x_l} +$$
$$+ \beta_1 F(x_1, x_2, t),$$
$$(x_1, x_2, t) \in \Omega \cap \{t > 0\}, \quad \Omega = \{(x_1, x_2): 0 < x_1 < 1, \quad -H/L < x_2 < 0\}, \tag{7}$$

$$c(0, x_2, t) = c_0(x_2, t), \quad -H/L \leqslant x_2 \leqslant 0, \tag{8}$$

$$\beta D_1 \frac{\partial c(x_1, x_2, t)}{\partial x_1}\bigg|_{x_1=1} = 0, \quad -H/L \leqslant x_2 \leqslant 0, \tag{9}$$

$$\beta D_2 \frac{\partial c(x_1, x_2, t)}{\partial x_2}\bigg|_{x_2=0} = \omega c(x_1, 0, t), \quad 0 \leqslant x_1 \leqslant L, \tag{10}$$

$$\beta D_2 \frac{\partial c(x_1, x_2, t)}{\partial x_2}\bigg|_{x_2=-\frac{H}{L}} = (1-\alpha)\,\omega c(x_1, -H/L, t), \quad 0 \leqslant x_1 \leqslant L, \tag{11}$$

$$c(x_1, x_2, 0) = \psi(x_1, x_2), \tag{12}$$

where $\beta = U_*/U_m$, $\beta_1 = L/(U_m c_1)$.

The results of [10] are then used to reduce (7)-(12) to minimization of

$$I(t) = \iint_{\Omega} \left\{ \sum_{l=1}^{2} \frac{\beta}{2} D_l \left(\frac{\partial c}{\partial x_l}\right)^2 + U_1 \frac{\partial c^0}{\partial x_1} c + \omega \frac{\partial c^0}{\partial x_2} c + \right.$$
$$\left. + \left(\frac{\partial c^0}{\partial t} - \beta_1 F\right) c \right\} dx_1 dx_2 + \frac{1}{2} \int_0^1 (1-\alpha)\,\omega c^2\left(x_1, -\frac{H}{L}, t\right) dx_1 - \frac{1}{2} \int_0^1 \omega c^2(x_1, 0, t)\, dx_1 \tag{13}$$

in a class of functions $c(x_1, x_2, t)$ that satisfy the conditions of (8) and (12); here superscript 0 denotes the desired variables, which are not used in variation in the local-potential method in (13).

In the finite-element method, we divide the region Ω into rectangular finite elements uniformly along the $0x_1$ and $0x_2$ direction by means of a grid having steps of Δ_1 and Δ_2, respectively. The solution is sought in the class of functions that satisfy (8):

$$\tilde{c}(x_1, x_2, t) = \sum_{i,j} C_{ij}(t)\, h_i\left(\frac{x_1}{\Delta_1}\right) h_j\left(\frac{x_2}{\Delta_2}\right), \tag{14}$$

where $h_i(x_1/\Delta_1)$, $h_j(x_2/\Delta_2)$ are finite piecewise-linear functions of the type

$$h_p\left(\frac{x_i}{\Delta_i}\right) = \frac{1}{2}\left(\left|\frac{x_i}{\Delta_i} - p - 1\right| - 2\left|\frac{x_i}{\Delta_i} - p\right| + \left|\frac{x_i}{\Delta_i} - p + 1\right|\right), \tag{15}$$

for which the following inequality can be proved [11]:

$$\|c - \tilde{c}\|_{L_2(\Omega)} \leqslant K(t)\,\Delta^2,$$

where K(t) is a function independent of Δ, and $\Delta = \max(\Delta_1, \Delta_2)$; note that $\tilde{c}(x_1, x_2, t)$ has the following features:

$$\tilde{c}(x_{1p}, x_{2q}, t) = C_{pq}(t), \quad \forall (x_{1p}, x_{2q}) \in \bar{\Omega},\ x_{i_p} = p\Delta_i, \quad i = 1, 2.$$

The functions $U_1(x_1, x_2, t)$, $D_i(x_1, x_2, t)$, i = 1, 2, $F(x_1, x_2, t)$ are represented in the same form as the function $c(x_1, x_2, t)$:

$$\tilde{U}_1(x_1, x_2, t) = \sum_{p,q} U_{pq}^{(1)}(t)\, h_p\left(\frac{x_1}{\Delta_1}\right) h_q\left(\frac{x_2}{\Delta_2}\right),$$

$$\tilde{D}_i(x_1, x_2, t) = \sum_{p,q} D_{pq}^{(i)}(t)\, h_p\left(\frac{x_1}{\Delta_1}\right) h_q\left(\frac{x_2}{\Delta_2}\right), \quad i = 1, 2, \tag{16}$$

$$\tilde{F}(x_1, x_2, t) = \sum_{p,q} f_{pq}(t)\, h_p\left(\frac{x_1}{\Delta_1}\right) h_q\left(\frac{x_2}{\Delta_2}\right).$$

We substitute (14) and (16) into (13) and equate the derivatives $\partial I(t)/\partial C_{\mu\nu}(t)$, $\mu = \overline{1, N}$, $\nu = \overline{1, M}$ to zero to get a Cauchy problem for the system of ordinary differential equations:

$$A\,\frac{d\mathbf{C}(t)}{dt} + B\mathbf{C}(t) = \mathbf{\Phi}(t), \tag{17}$$

where $\underline{C}(t) = (C_{11}(t), \ldots, C_{1N}(t), \ldots, C_{M1}(t), \ldots, C_{MN}(t))^T$, $C_{pq}(0) = \psi(x_{1p}, x_{2q})$, $\forall (x_{1p}, x_{2q}) \in \Omega$; here A and B are matrices of dimensions MN whose elements are, respectively,

$$a_{\mu+(\nu-1)N,\, i+(j-1)N} = \iint_\Omega \Theta_{ij}(x_1, x_2)\,\Theta_{\mu\nu}(x_1, x_2)\, dx_1 dx_2,$$

$$b_{\mu+(\nu-1)N,\, i+(j-1)N} = \sum_{p,q}\left\{\sum_{l=1}^{2} \beta D_{pq}^{(l)}(t) \iint_\Omega \frac{\partial\Theta_{ij}(x_1, x_2)}{\partial x_l}\,\Theta_{pq}(x_1, x_2) \times\right.$$

$$\times \frac{\partial\Theta_{\mu\nu}(x_1, x_2)}{\partial x_l}\, dx_1 dx_2 + U_{pq}^{(1)}(t) \iint_\Omega \frac{\partial\Theta_{ij}(x_1, x_2)}{\partial x_1}\,\Theta_{pq}(x_1, x_2) \times$$

$$\left.\times\,\Theta_{\mu\nu}(x_1, x_2)\, dx_1 dx_2\right\} + \omega\left\{\iint_\Omega \frac{\partial\Theta_{ij}(x_1, x_2)}{\partial x_2}\,\Theta_{\mu\nu}(x_1, x_2)\, dx_1 dx_2 + [\delta_{\nu 1}(1-\alpha) - \delta_{\nu M}] \int_0^1 h_i\left(\frac{x_1}{\Delta_1}\right) h_\mu\left(\frac{x_1}{\Delta_1}\right) dx_1\right\};$$

and $\Phi(t)$ is a column vector having the components

$$\varphi_{\mu+(\nu-1)N} = \sum_{p,q} \beta_1 f_{pq}(t) \iint_\Omega \Theta_{pq}(x_1, x_2)\,\Theta_{\mu\nu}(x_1, x_2)\, dx_1 dx_2,$$

$$\Theta_{ij}(x_1, x_2) = h_i\left(\frac{x_1}{\Delta_1}\right) h_j\left(\frac{x_2}{\Delta_2}\right), \quad \mu = \overline{1, N}, \ \nu = \overline{1, M},$$

$$i, p = \overline{\mu - 1 + \delta_{\mu 1}, \mu + 1 - \delta_{\mu N}}, \quad j, q = \overline{\nu - 1 + \delta_{\nu 1}, \nu + 1 - \delta_{\nu M}};$$

and C(t) is the column vector for the unknown functions. A novel algorithm for incorporating (8) was used in solving (17).

An Algol program for the BESM-6 has been written to solve (7)-(12).

Various difficulties arise in solving this type of problem [15], which cannot be discussed here; a standard procedure (RUKUT) was used in solving (17) for integrating ordinary differential equations by the Runge–Kutta method, with automatic step-size choice. The choice is designed to provide a reasonably stable computation, and the results confirm that this is so.

The following are some numerical results.

Example 1. The algorithm was checked out by solving (7)-(12) with

$$\beta_1 F = \lambda(1-x_1)\exp(-\lambda t) - U_1[1-\exp(-\lambda t)],$$

$$c_0(x_2, t) = 1-\exp(-\lambda t),$$

$$c(1, x_2, t) = 0, \quad \beta D_2 \frac{\partial c}{\partial x_2}\Big|_{x_2=0} = 0, \quad \beta D_2 \frac{\partial c}{\partial x_2}\Big|_{x_2=-H/L} = 0,$$

and $\psi(x_1, x_2) = 0$, which has the exact solution $c = (1-x_1)[1-\exp(-\lambda t)]$; in that case, the steady-state longitudinal velocity distribution is given in [14, p. 100], viz., U_{av} = 0.02 m/sec, U_m = 0.025 m/sec, where D_1 and D_2 are constants: $D_1 = 0.4\cdot 10^{-5}$, $D_2 = 0.1237\cdot 10^{-5}$. Table 1 shows that the approximate solution is essentially the same as the accurate one. The run time was 8 min.

Example 2. Problem (7)-(12) was solved for L = 1000 m, H = 1.75 m to give the longitudinal velocity distribution as in Example 1, with $\alpha = 0.5$, $-\omega = 0.00252$, $\psi(0, x_2) = 1$; $\psi(x_1, x_2) = 0$, $(x_1 > 0)$; the coefficients D_x and D_z were taken as the following constants [12,13]:

$$D_x = 0.22HU_*, \quad D_z = 0.068HU_*. \tag{18}$$

We also solve the stationary-state problem corresponding to (7)-(12), for which similar steps give the system of algebraic equations

$$B\mathbf{C} = \mathbf{\Phi}.$$

If t is sufficiently large (t = 2), the solution to the nonstationary problem approaches that for the stationary one (Table 2). The run time was 4 min for the stationary problem with the region divided into 72 finite elements (91 nodes), as against 8 min for the nonstationary case.

Example 3. The problem of [3] was solved for the stationary case in order to check the algorithm. Figure 1 shows that our finite-element solution agrees with the finite-difference solution of [3] (the origin was set at the bottom, $x_2 = z/H$ in [3]). The model used for the steady state in Example 2 was also checked by using the constant coefficients of (18), as well as with

$$D_x = 0.22HU_*, \quad D_z = kU_*z\left(1+\frac{z}{H}\right). \tag{19}$$

Figure 1 shows the results. The solution with the coefficients of (19) coincides within the accuracy allowed by the graph with our finite-element solution of [3], which indicates that the D_z distribution derived by experiment in [3] is only slightly better than that provided by (19). Figure 1 also shows that the D_z averaged over the depth of (18) gives entirely satisfactory results. All the results obtained in this example were with Ω divided into 100 finite elements (124 nodes). The run time for one form was 5 min.

Example 4. Finally we give results from numerical solution of (7)-(12) with $c_0(x_2, t) = \exp(-\lambda t)$ and with the U_1 and coefficients D_i, i = 1, 2, and initial conditions of Example 2, with $\alpha = 0.5$; Fig. 2 gives the results.

The model has been checked out on experimental data (Example 3), because the algorithm was written for a reasonably general case (variable D_x and D_z, various U and W as functions of time and coordinate), so it can be used in numerous practical instances. For instance, Examples 2 and 4 are derived from attempts to forecast the distribution and deposition of suspended matter in a fairly broad channel or extended pond. Solutions to such problems indicate ways of preventing silting-up in irrigation systems and reservoirs. In hydroelectric power engineering, the content of solid matter in the water has a considerable effect on the

TABLE 1. Lengthwise Concentration Distribution (Example 1), $\lambda = 10$ and $t = 1$

x_1	0	1/6	2/6	3/6	4/6	5/6	1
Approximate solution	0,9999546	0,8332941	0,6666438	0,4999802	0,3333269	0,1666570	0
Exact	0,9999546	0,8332955	0,6666364	0,4999773	0,3333182	0,1666591	0

TABLE 2. Results from Numerical Solution of the Nonstationary Problem (A) and Stationary Problem (B) of Example 2

Type	x_2	x_1						
		0	1/6	2/6	3/6	4/6	5/6	1
A	0	1	0,3172	0,1809	0,1171	0,0805	0,0569	0,0410
B	0	1	0,3151	0,1786	0,1163	0,0801	0,0564	0,0401
A	$-H/6L$	1	0,5159	0,3034	0,1996	0,1380	0,0972	0,0686
B		1	0,4959	0,2993	0,1983	0,1374	0,0970	0,0689
A	$-H/3L$	1	0,6989	0,4482	0,3062	0,2151	0,1530	0,1096
B		1	0,6894	0,4449	0,3048	0,2142	0,1520	0,1083
A	$-H/2L$	1	0,8494	0,6145	0,4388	0,3136	0,2239	0,1594
B		1	0,8511	0,6120	0,4377	0,3125	0,2231	0,1594
A	$-2H/3L$	1	0,9918	0,8054	0,5996	0,4353	0,3131	0,2247
B		1	1,0027	0,8059	0,5990	0,4338	0,3114	0,2230
A	$-5H/6L$	1	1,1911	1,0387	0,7929	0,5805	0,4182	0,2990
B		1	1,2093	1,0435	0,7927	0,5789	0,4166	0,2984
A	$-H/L$	1	1,5460	1,3636	1,0454	0,7667	0,5532	0,3970
B		1	1,5558	1,3688	1,0453	0,7642	0,5504	0,3943

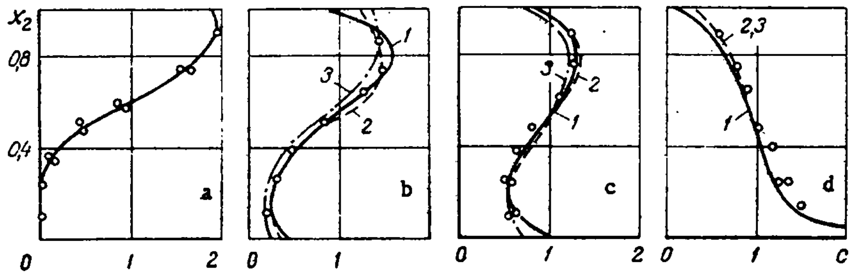

Fig. 1. Depth distribution of suspended matter (Example 3): a) at inlet ($x_1 = 0$); b-d) for $x_1 = 0.1351$, 0.2696, 0.6804, respectively; the points are from experiment [3]; 1) finite-difference solution [3]; 2, 3) our solution (D_x and D_z in accordance with (18)).

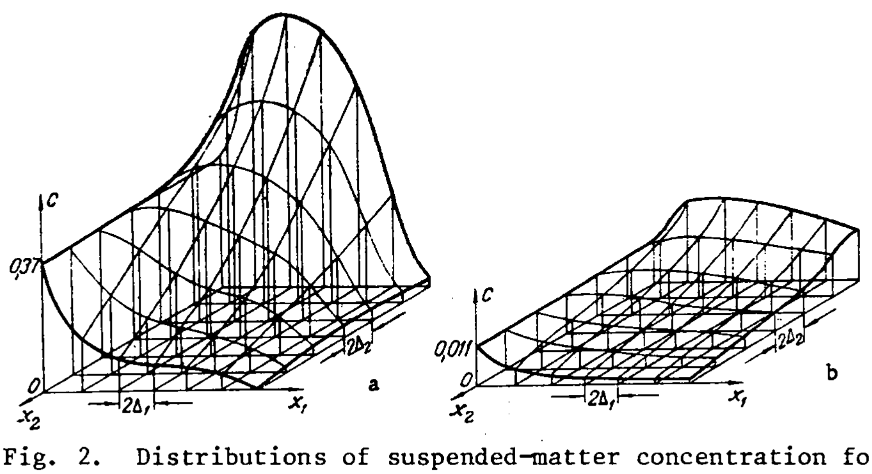

Fig. 2. Distributions of suspended-matter concentration for $t = 1$ (a) and $t = 4.5$ (b) (Example 4).

working life of the turbines, and here the prediction of the behavior of solid material is particularly important.

NOTATION

Ω_1, domain of spatial variables with boundary Γ; Γ_1 and Γ_3, boundary segments parallel to Oz (inlet and outlet, respectively); Γ_2, free surface; Γ_4, bottom of contour; n, direction of the exterior normal to the boundary of Ω_1; L and H, channel length and depth; x and z, horizontal and vertical coordinates; t', time; U and W, horizontal and vertical velocity components; c', impurity concentration; ω', hydraulic parameter; D_x and D_z, turbulent-diffusion coefficients; F', source (sink) function; $0 \le \alpha \le 1$, bottom-absorption coefficient; U_*, dynamic velocity, U_m, free-surface value of U; U_{av}, average value of U; c_1, characteristic inlet value of impurity concentration; $\Delta_1(\Delta_2)$, step size along $Ox_1(Ox_2)$; δ_{ij}, Kronecker symbol; k, Karman constant.

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CHARACTERISTIC METHOD IN HEAT TRANSPORT IN FAST NONSTATIONARY PROCESSES

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UDC 536.2

The characteristic method has been used in numerical solution of a hyperbolic heat-transport equation.

The following hyperbolic equation is involved in heat-transfer calculation for fast nonstationary processes of one-dimensional type:

$$c\rho \frac{\partial T}{\partial \tau} + c\rho\tau_r \frac{\partial^2 T}{\partial \tau^2} = \lambda \frac{\partial^2 T}{\partial x^2} \qquad (1)$$

subject to the appropriate initial and boundary conditions. As a rule, the boundary conditions are nonlinear, and then there are major difficulties in obtaining an analytic solution. A network method (explicit difference scheme) has been used [1] to solve (1). Studies have been made [2,3] on the construction of difference schemes for equations of hyperbolic type on the basis of characteristic relationships, particularly with regard to the stability; here we show that the characteristic method can be applied in heat-transfer calculations for fast nonstationary processes.

We first put

$$V = \frac{\partial T}{\partial \tau}, \quad W = \frac{\partial T}{\partial x}, \qquad (1)$$

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